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5D Generalized Inflationary Cosmology

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Abstract

We consider 5D Kaluza-Klein type cosmological model with the fifth coordinate being a generalization of the invariant "historical" time τ of the covariant theory of Horwitz and Piron. We distinguish between vacuum-, off-shell matter-, and on-shell matter-dominated eras as the solutions of the corresponding 5D gravitational field equations, and build an inflationary scenario according to which passage from the off-shell matter-dominated era to the on-shell one occurs, probably as a phase transition. We study the effect of this phase transition on the expansion rate in both cases of local $O(4,1)$ and $O(3,2)$ invariance of the extended (x^μ,τ) manifold and show that it does not change in either case. The expansion of the model we consider is not adiabatic; the thermodynamic entropy is a growing function of cosmic time for the closed universe, and can be a growing function of historical time for the open and the flat universe. A complete solution of the 5D gravitational field equations is obtained for the on-shell matter-dominated universe. The open and the closed universe are shown to tend asymptotically to the standard 4D cosmological models, in contrast to the flat universe which does not have the corresponding limit. Finally, possible cosmological implications are briefly discussed.

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1 Introduction

The possibility that space-time has more than four dimensions has received much attention regarding its cosmological aspects [1–8]. Investigations have focused on attempts to explain why the universe presently appears to have only four space-time dimensions if it is, in fact, a dynamically evolving $(4+k)$-dimensional manifold (k being the number of extra dimensions). It has been shown that solutions to the $(4+k)$-dimensional Einstein equations exist, for which four-dimensional space-time expands while the extra dimensions contract or remain constant [4–8]. It has been pointed out that the extra dimensions can produce large amounts of entropy during the contraction process [6], thus providing an alternative resolution to the horizon and flatness problems [3], as compared to the usual inflationary scenario. It has been also suggested that experimental detection of the time variation of the fundamental constants could provide strong evidence for the existence of extra dimensions [8].

In the present paper we study the 5D cosmological model of Kaluza-Klein type, with the fifth coordinate being a generalization of the universal parametric "historical" time τ discussed by, for example, Stueckelberg [9] and Horwitz and Piron [10]. It has been shown that gauge invariance of the Stueckelberg-Schrödinger equation requires the addition of a fifth gauge field [11]; this result also follows from Feynman's approach to the foundations of gauge theories in a manifestly covariant framework [12]. The equations of motion for such a gauge field are of second order in the five-dimensional manifold (x^μ, τ), with metric (4,1) or (3,2); i.e., on the level of the gauge fields, the parametric "historical" time has entered a five-dimensional manifold, much in the way that the Newtonian time t enters the four-dimensional Minkowski manifold as a consequence of the requirements of full gauge invariance of the Schrödinger equation. The canonical quantization of this $U(1)$ gauge theory has been carried out by Shnerb and Horwitz [13], where it is shown that the standard Maxwell theory is recovered in a "correlation" limit.

The present paper is concerned with a 5D theory of gravitation with τ as a fifth coordinate, which originated in an earlier work [14], in which we considered the thermodynamics of a relativistic N-body system, taking account of the mass distribution in such a system [15, 16]. In [14] we incorporated two-body interactions, by means of the direct action potential $V(q)$, where q is an invariant distance in the Minkowski space, taking the support of the two-body correlations to be in a $O(2,1)$ invariant subregion of the full spacelike region of relative coordinates. We then established the energy conditions on matter in order that the Einstein equations possess a singularity in terms of $V(q)$, and showed that, for a class of power-law attractive potentials,

$V(q) \sim q^n,\ 0 < n \leq 3$, the energy conditions for a singularity to occur can be violated only in the case of local $O(3,2)$ invariance of the (x^μ, τ) manifold.

We have found that an off-shell ensemble at high temperatures is characterized by the equation of state $p = (\Gamma - 1)\rho;\quad p, \rho \propto T^{\Gamma/(\Gamma-1)}$, with Γ being equal to $3/2$ in the case of local $O(3,2)$ invariance of the (x^μ, τ) manifold[1] $(\sigma = 1)$ and $5/4$ in the case of local $O(4,1)$ invariance $(\sigma = -1)$, so that in the latter case $p = \rho/4,\quad p, \rho \propto T^5$.

Off-shell matter[2] with the equation of state $p, \rho \propto T^5$ was introduced into the standard cosmological model in ref. [18]. It was shown that such matter has energy density comparable with that of standard radiation (with the equation of state $p, \rho \propto T^4$) at temperature $\sim 10^{12}$ K, so that the possibility for a phase transition from the off-shell sector to the on-shell one (with possible compactification of the fifth dimension [19]), at critical temperature $\sim 10^{12}$ K, should be taken into account; for example, in the case of a Bose gas, by the mechanism of a high-temperature Bose-Einstein condensation [20].

As we show in the present paper, a 5D cosmological model of Kaluza-Klein type permits derivation of vacuum-, off-shell matter-, and on-shell matter-dominated eras as the solutions of the corresponding 5D gravitational field equations. These solutions enable one to construct an inflationary scenario (inflationary solutions arise in a vacuum-dominated era) according to which, as the universe expands and cools down, a phase transition from the off-shell sector to the on-shell one occurs, probably at temperature $\sim 10^{12}$ K [18]. We study its effect on the rate of expansion and show that in both cases of $(\sigma = -1) \rightarrow (\sigma = 0)$ and $(\sigma = 1) \rightarrow (\sigma = 0)$ phase transition[3] the expansion rate does not change.

We show that the model we are discussing does not expand adiabatically. For the closed universe the thermodynamic entropy is a growing function of cosmic time; for the flat and the open universe it can be a growing function of historical time. The open and the closed models will be shown to go to the 4D standard cosmological models as the universe expands, in contrast to the flat model which does not have the corresponding limit.

We remark that some previous discussions of these questions have been made in the framework of 5D Kaluza-Klein theory [21]–[26]. Mann and Vincent [21] have shown that the vacuum (Kaluza-Klein type) solutons of the five-dimensional field equations give rise to an effective radiation density $(\rho = 3p)$ connected with the extra dimension. Ponce de Leon and Wesson [26] have interpreted the sourceless solutions of the five-dimensional Kaluza-Klein equations as those of the four-dimensional Einstein equations with effective matter properties. We also note that inflationary models based on the Kaluza-Klein framework have been considered by Shafi and Wetterich

[1]The denotement σ stands for the 55-component of the local metric on the (x^μ, τ) manifold which is $g^{\alpha\beta} = (+,-,-,-,\sigma)$.

[2]We use the term 'off-shell' to describe matter with continuous mass distribution, i.e., non-point spectrum, as for off-shell states occuring in the propagators of quantum field theory.

[3]We use $\sigma = 0$ to describe the standard (on-shell) 3+1 case.

[27], and by Grøn [28] who has derived a complete cosmological scenario within the framework of Wesson's gravitational theory with the rest mass as a fifth coordinate [29].

2 The line element

Similarly to [14], we take the fifth-dimension subspace to be homogeneous and without coupling to the other coordinate, i.e., a maximally symmetric subspace of the 5D space [30]. Then the 5D metric becomes [30]

$$ {}^{(5)}ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g_{\mu\nu}(x^\rho) dx^\mu dx^\nu + g_{55}(x^\rho) d\tau^2, \tag{2.1} $$

$$ \alpha, \beta = 0, 1, 2, 3, 5; \quad \mu, \nu, \rho = 0, 1, 2, 3. $$

As shown in Appendix, the 5D gravitational field equations

$$ {}^{(5)}R_{\alpha\beta} = 8\pi G \left({}^{(5)}T_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} \, {}^{(5)}T^\lambda_\lambda \right), \tag{2.2} $$

with the source term

$$ {}^{(5)}T_{\alpha\beta} = \left({}^{(4)}T_{\mu\nu}, \; p_5 \right), \quad p_5 = \sigma \mu_K \kappa, \tag{2.3} $$

where κ is the density of the generalized Hamiltonian per unit comoving three-volume (actually associated with the density of the variable mass) and μ_K is the mass potential in relativistic ensemble [17], reduce to the 4D Einstein equations

$$ {}^{(4)}R_{\mu\nu} = 8\pi G \left({}^{(4)}T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, {}^{(4)}T^\rho_\rho \right) \tag{2.4} $$

in the case of no curvature in τ direction.

As an example of the use of the metric (2.1) in cosmology, consider a spacially flat cosmological model with the line element (in refs. [21, 25, 26] a similar structure has been used)

$$ ds^2 = e^{\bar{\nu}} dt^2 - e^{\bar{\omega}}(dx^2 + dy^2 + dz^2) - e^{\bar{\mu}} d\tau^2, \tag{2.5} $$

where $\bar{\nu}, \bar{\omega}, \bar{\mu}$ are assumed (here) to be functions of time alone. A particular solution is obtained for $\bar{\nu} = 0$, $e^{\bar{\omega}} = t$, $e^{\bar{\mu}} = t^{-1}$. In this case the line element

$$ ds^2 = dt^2 - t(dx^2 + dy^2 + dz^2) - t^{-1} d\tau^2 \tag{2.6} $$

is similar to the cosmological model found by Chodos and Detweiler [2]. They interpreted the fifth dimension geometrically in the usual Kaluza-Klein sense [31]. The time coordinate of the line element (2.6) is the proper time shown on standard clocks at rest in the 3D spacial hyperplane orthogonal to the time- and τ-directions. This

will in the following be referred to as "cosmic time". In the case that 4D space-time is filled with a medium in which these clocks are at rest, the coordinate system is said to be "comoving". These are the usual terms from ordinary 4D cosmology. The expansion factor of the model (2.6) is $R(t) = t^{1/2}$. This universe expands too slowly to solve the horizon and flatness problems [3].

By the proper choice of s, the line element (2.5) can be reduced to

$$ds^2 = dt^2 - e^{\omega}(dx^2 + dy^2 + dz^2) - e^{\mu} d\tau^2. \tag{2.7}$$

For this line element, the nonvanishing Christoffel symbols are (henceforth the prime stands for derivative with respect to the 5D line element, and the dot for derivative with respect to cosmic time)

$$\Gamma^5_{05} = \frac{\dot{\mu}}{2}, \quad \Gamma^0_{55} = \frac{1}{2}(\dot{e^{\mu}}). \tag{2.8}$$

Therefore, the geodesic equations for t and τ read (see Appendix)

$$t^{''} + \frac{1}{2}(\dot{e^{\mu}})(\tau^{'})^2 = 0, \tag{2.9}$$

$$\tau^{''} + \dot{\mu} t^{'} \tau^{'} = 0. \tag{2.10}$$

The Lagrangian of a free comoving particle is

$$L = e^{\mu}(\tau^{'})^2 - (t^{'})^2. \tag{2.11}$$

Since τ is a cyclic coordinate (it does not appear in the Lagrangian), the conjugate momentum

$$p_\tau = \frac{\partial L}{\partial \tau^{'}} = e^{\mu}\tau^{'} \tag{2.12}$$

is a constant of motion, giving

$$\frac{d\tau}{ds} = p_\tau e^{-\mu}. \tag{2.13}$$

Inserting (2.13) into Eq. (2.9) gives

$$t^{''} = -\frac{p_\tau^2}{2} e^{-\mu}\dot{\mu} = \frac{p_\tau^2}{2}(e^{-\mu})^{'}(t^{'})^{-1}. \tag{2.14}$$

Integration leads to

$$\frac{ds}{dt} = \left(p_\tau^2 e^{-\mu} + C^2\right)^{-1/2}, \tag{2.15}$$

where C is an arbitrary constant. It then follows from (2.13),(2.15) that

$$\frac{d\tau}{dt} = \frac{p_\tau e^{-\mu}}{(p_\tau^2 e^{-\mu} + C^2)^{1/2}}. \tag{2.16}$$

We now shall consider the following generalization of the line element (2.5) which lets a spacial curvature be different from zero and permits direct comparison with the Kaluza-Klein models[4]:

$$ds^2 = dt^2 - \frac{R^2(t)}{(1+\frac{1}{4}kr^2)^2}(dx^2+dy^2+dz^2) - A^2(t)d\tau^2, \tag{2.18}$$

where $r^2 = x^2+y^2+z^2$, and $k = 0, \pm 1$ characterizes the spacial curvature. Comparison with Eq. (2.7) shows that $e^\mu = A^2$, so that, expressed in terms of A, Eqs. (2.13), (2.16) take on the form

$$\frac{ds}{d\tau} = \frac{A^2}{p_\tau}, \tag{2.19}$$

$$\frac{ds}{dt} = (p_\tau^2/A^2 + C^2)^{-1/2}, \tag{2.20}$$

$$\frac{dt}{d\tau} = \frac{(p_\tau^2/A^2 + C^2)^{1/2}}{p_\tau/A^2}. \tag{2.21}$$

Since the standard case of no curvature in τ direction corresponds to $A = \text{const}$, in this case $ds/d\tau = \text{const}$, so that the line element reduces (as seen in Eq. (2.17),(2.18); see also Appendix) to

$$d\tau^2 = dt^2 - \frac{R^2(t)}{(1+\frac{1}{4}kr^2)^2}(dx^2+dy^2+dz^2), \tag{2.22}$$

which in turn can be reduced to the standard 4D Robertson-Walker metric

$$d\tau^2 = dt^2 - R^2(t)\left[\frac{d\rho^2}{1-k\rho^2} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2.23}$$

with the help of the transformation [32]

$$\rho = \frac{r}{1+\frac{1}{4}kr^2}. \tag{2.24}$$

We see that, as $A \to \text{const}$, the 5D universe with the line element (2.17),(2.18) passes over to the standard 4D Robertson-Walker universe[5] (a phase transition to the on-shell sector at $T \sim 10^{12}$ K probably taking place).

[4]The 5D metric (2.18) corresponds to local $O(4,1)$ invariance of an extended (x^μ, τ) manifold. The choice of the metric in the form

$$ds^2 = dt^2 - \frac{R^2(t)}{(1+\frac{1}{4}kr^2)^2}(dx^2+dy^2+dz^2) + A^2(t)d\tau^2, \tag{2.17}$$

corresponding to local $O(3,2)$ invariance of an (x^μ, τ) manifold, will lead essentially to the results of this work. Both cases (2.17),(2.18) are treated simultaneously in the system of the field equations (3.2)-(3.4).

[5]We remark that the universe with the Robertson-Walker type metric

$$ds^2 = d\tau^2 - R^2(\tau)\left[\frac{d\rho^2}{1-k\rho^2} + \rho^2\Big(d\theta^2 + \sin^2\theta(-d\beta^2 + \cosh^2\beta d\phi^2)\Big)\right],$$

3 The field equations

We consider the following field equations for a 5D space-time filled with a perfect fluid, permitting a non-vanishing cosmological constant (henceforth we shall use the system of units in which $c = 8\pi G = 1$) :

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = T_{\alpha\beta}, \tag{3.1}$$

with the source term (2.3). We note that the cosmological constant could be disposed of by considering instead space-time with a vacuum fluid which is a perfect fluid with the equation of state $p = -\rho$.

The field equations (3.1) with $T_{\alpha\beta}$ in the form (2.3) reduce to the following system [6, 21]:

$$\frac{\dot{R}^2 + k}{R^2} + \frac{\dot{R}\dot{A}}{RA} = \frac{1}{3}(\Lambda + \rho), \tag{3.2}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} + \frac{2\dot{R}\dot{A}}{RA} + \frac{\ddot{A}}{A} = \Lambda - p, \tag{3.3}$$

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2 + k}{R^2} = \frac{1}{3}(\Lambda + p_5). \tag{3.4}$$

The usual 4D equations of the Friedmann model are obtained by setting $A = \text{const}$ in Eqs. (3.2),(3.3) and neglecting Eq. (3.4). The energy-momentum conservation $T^{\alpha\beta}_{;\beta} = 0$ implies [6, 21]

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} + (\rho - p_5)\frac{\dot{A}}{A} = 0. \tag{3.5}$$

For the initial stage of the evolution, when the universe is hot, we can use [14]

$$p_5 = \sigma p. \tag{3.6}$$

This result, in fact, follows easily from the definition of the five-dimensional energy-momentum tensor (also discussed in [14]) which is obtained by the extension of the usual energy-momentum tensor

$$T^{\mu\nu} = (p + \rho)u^\mu u^\nu - p g^{\mu\nu}, \quad u^\rho u_\rho = 1 \tag{3.7}$$

to a five-dimensional form:

$$^{(5)}T_{\alpha\beta} = \left({}^{(5)}T_{\mu\nu},\ {}^{(5)}T_{55} \right); \tag{3.8}$$

where $\rho, \theta, \beta, \phi$ are the coordinates in the restricted Minkowski space (RMS) [14], as $R(\tau) \to \infty$, passes over to the standard 4D Robertson-Walker universe as well. Details will be explained elsewhere [33].

the requirement that the limiting case of the corresponding gravitational theory (for zero curvature in the τ direction) coincides with the Einstein equations results in the identification (see Appendix) ${}^{(5)}T_{\mu\nu} = {}^{(4)}T_{\mu\nu}$ and ${}^{(5)}T_{55} = \sigma\mu_K\kappa$. Expressions for p and ρ, using the grand canonical ensemble obtained by Horwitz, Schieve and Piron [17] in their study of manifestly covariant statistical mechanics, were found in [16] in terms of confluent hypergeometric functions. For T small, one finds that $p, \rho \propto T^6$, $\rho \simeq 5p$, and, in fact, that $\mu_K\kappa \propto T^7$ is negligible in comparison with ρ. On the other hand, for T large, one finds [14] that in the case of local $O(4,1)$ invariance of (x^μ, τ) manifold, $p, \rho \propto T^5$, $p \simeq \rho/4 \simeq \mu_K\kappa$, while in the case of local $O(3,2)$ invariance, $p, \rho \propto T^3$, $p \simeq \rho/2 \simeq \mu_K\kappa$. For high temperature, it therefore follows that (as discussed in [14])

$$T^{\alpha\beta} = (p+\rho)u^\alpha u^\beta - pg^{\alpha\beta}, \quad u^\lambda u_\lambda = 1, \tag{3.9}$$

so that, in the local rest frame, $T_{\alpha\beta} =$ diag $(\rho, -p, -p, -p, \sigma p)$, and Eq. (3.6) is justified. Moreover, for a perfect fluid, we use the equation of state

$$p = (\Gamma - 1)\rho, \tag{3.10}$$

where Γ is a constant. It then follows [14] that in the cases of $\sigma = -1$ ($O(4,1)$) and $\sigma = 1$ ($O(3,2)$), Γ is equal to $5/4$ and $3/2$, respectively, so that the tensor (3.9) is traceless in either case. We note that the form (3.9) of a source term for the 5D field equations has been used by Wesson [29].

In the case of local $O(4,1)$ invariance of an (x^μ, τ) manifold, $\sigma = -1$, $p_5 = -p$, and Eq. (3.5) integrates to

$$R^{3\Gamma}A^{2-\Gamma}\rho = \text{const.} \tag{3.11}$$

The expansion is not adiabatic. As $A \to$ const, Eq. (3.11) takes on the standard form

$$R^{3\Gamma}\rho = \text{const.} \tag{3.12}$$

Similarly, in the case of local $O(3,2)$ invariance of an (x^μ, τ) manifold, $\sigma = 1$, $p_5 = p$, and Eq. (3.5) gives

$$R^{3\Gamma}A^{\Gamma}\rho = \text{const,} \tag{3.13}$$

wich again reduces to the standard form (3.12) as $A \to$ const. Note that both Eqs. (3.11) and (3.13) give for dust matter with $p \approx 0$ ($\Gamma = 1$) (this can also be obtained directly from (3.5) with $p = p_5 = 0$)

$$R^3 A\rho = \text{const,} \tag{3.14}$$

which, as $A \to$ const, reduce to the standard result

$$R^3\rho = \text{const.} \tag{3.15}$$

4 Solutions to the field equations

The standard strategy for solving cosmological equations, i.e., to exclude ρ and p from the equation for R in terms of ρ, p, with the help of the equations of state and energy-momentum conservation: $p = (\Gamma - 1)\rho$, $\rho \propto R^{-n}$, $n = 3, 4$, does not work in our case, since, in view of (3.11),(3.13), $\rho \propto R^{-p}A^{-q}$ and A is a function of cosmic time. We therefore have to express R in terms of a parameter which is independent of A. Such a parameter is a cosmological constant Λ.

We shall suppose that A is a slowly varying function of t, so that one can neglect the term $\ddot{A}/A$ in Eq. (3.3). Then, for $\sigma = -1$ ($O(4,1)$), one derives from (3.2)-(3.4), by the exclusion of ρ, p and p_5 with the help of (3.6),(3.10), the equation

$$\frac{\Gamma + 1}{2(2\Gamma - 1)}\ddot{R}R + \dot{R}^2 + k = \frac{\Gamma}{3(2\Gamma - 1)}\Lambda R^2, \tag{4.1}$$

which for $\Gamma = 5/4$ reduces to

$$\frac{3}{4}\ddot{R}R + \dot{R}^2 + k = \frac{5}{18}\Lambda R^2. \tag{4.2}$$

For $\sigma = 1$ ($O(3,2)$) one similarly obtains

$$\frac{5\Gamma - 3}{2\Gamma}\ddot{R}R + \dot{R}^2 + k = \frac{2\Gamma - 1}{3\Gamma}\Lambda R^2, \tag{4.3}$$

which for $\Gamma = 3/2$ reduces to

$$\frac{3}{2}\ddot{R}R + \dot{R}^2 + k = \frac{4}{9}\Lambda R^2. \tag{4.4}$$

For the standard case of $p_5 = 0$ (or $\sigma = 0$) one gets

$$\ddot{R}R + \dot{R}^2 + k = \frac{1}{3}\Lambda R^2, \tag{4.5}$$

which really represents Eq. (3.4) with $p_5 = 0$. Note that both Eqs. (4.1) and (4.3) reduce to (4.5) for $\Gamma = 1$.

4.1 Vacuum-dominated era

In a vacuum-dominated era the universe is filled with a vacuum fluid. Eqs. (4.2),(4.4), (4.5) can be represented by an equation of the general form

$$a\ddot{R}R + \dot{R}^2 + k = b\Lambda R^2, \tag{4.6}$$

which, through the substitution

$$R^{1+\frac{1}{a}} = \tilde{R}, \tag{4.7}$$

reduces to the equation

$$\ddot{\tilde{R}} - \frac{b(a+1)}{a^2}\Lambda \tilde{R} + \frac{a+1}{a^2} k \tilde{R}^{\frac{1-a}{1+a}} = 0 \tag{4.8}$$

having the solution ($C_1, C_2 = \mathrm{const}$)

$$\tilde{R} = C_1 \cosh \sqrt{\frac{b(a+1)}{a^2}\Lambda}\, t + C_2 \sinh \sqrt{\frac{b(a+1)}{a^2}\Lambda}\, t + \left(\frac{k}{b\Lambda}\right)^{\frac{1}{2}(1+\frac{1}{a})}. \tag{4.9}$$

We, therefore, obtain from (4.2),(4.4),(4.5), respectively:

for $\sigma = -1$,

$$R^{7/3} = \left(\frac{18}{5}\frac{k}{\Lambda}\right)^{7/6} + C_1^- \cosh \frac{\sqrt{70\Lambda}}{9} t + C_2^- \sinh \frac{\sqrt{70\Lambda}}{9} t, \tag{4.10}$$

for $\sigma = 1$,

$$R^{5/3} = \left(\frac{9}{4}\frac{k}{\Lambda}\right)^{5/6} + C_1^+ \cosh \frac{\sqrt{40\Lambda}}{9} t + C_2^+ \sinh \frac{\sqrt{40\Lambda}}{9} t, \tag{4.11}$$

for $\sigma = 0$,

$$R^2 = \frac{3k}{\Lambda} + C_1^0 \cosh \frac{\sqrt{54\Lambda}}{9} t + C_2^0 \sinh \frac{\sqrt{54\Lambda}}{9} t. \tag{4.12}$$

We note that the solution (4.12) was obtained previously by Grøn [28]. In subsequent consideration we shall, for simplicity, restrict ourselves to this solution alone. It then follows that, without any loss of generality, this solution can be represented by the following relations:

$k = 1$,

$$R^2 = \frac{2}{\omega^2}(1 + \cosh \omega t) = \frac{4}{\omega^2}\cosh^2 \frac{\omega t}{2}, \tag{4.13}$$

$k = -1$,

$$R^2 = \frac{2}{\omega^2}(\cosh \omega t - 1) = \frac{4}{\omega^2}\sinh^2 \frac{\omega t}{2}, \tag{4.14}$$

$k = 0$,

$$R^2 = \frac{4}{\omega^2}\exp{(\omega t)}, \quad w \equiv \sqrt{\frac{2}{3}\Lambda}. \tag{4.15}$$

Consider, for example, (4.13). For $t = 0$ it gives $R = 2/\omega$. Since the classical description of the expansion of the universe cannot be valid prior to $t \sim t_{Pl} = M_{Pl}^{-1} \sim 5\cdot 10^{-44}$ s after the big bang or the start of inflation at $t = 0$, one finds that $\omega \lesssim M_{Pl} \sim 10^{19}$

GeV, and therefore $\Lambda = \frac{3}{2}\omega^2 \lesssim 10^{38}$ GeV^2. By introducing the vacuum energy density through the relation

$$\Lambda c^2 = 8\pi G \rho_{vac} \tag{4.16}$$

and recovering c and G for numerical calculation, $G \sim M_{Pl}^{-2} \simeq 10^{-38}$ GeV^{-2}, one obtains

$$\rho_{vac} < M_{Pl}^4/16 \sim 10^{75}\ \mathrm{GeV}^4 \simeq 10^{92}\ \mathrm{g\ cm}^{-3}. \tag{4.17}$$

If, similar to the standard inflationary models [34], one takes $\rho_{vac} = T_c^4 \sim 10^{60}$ GeV^4, where $T_c \sim 10^{15}$ GeV is a typical critical temperature for a phase transition in grand unified theories [34], one obtains $\omega = \sqrt{16\pi G \rho_{vac}/3c^2} \simeq 4 \cdot 10^{11}$ GeV. As is usually done in the standard inflationary models [34], inflation comes to an end when its rate $H \equiv \dot{R}/R = \omega/2$ begins to decrease rapidly (which means that the universe becomes rapidly increasing in size), the typical time of inflation is $t_{inf} \sim 1/H = 2/\omega$. With $\omega \simeq 4 \cdot 10^{11}$ GeV, one finds $t_{inf} \simeq 10^{-36}$ s.

The value of the vacuum energy density (4.16) should be related to the present-day vacuum energy density which is not much greater in absolute value than the critical density $\rho_{cr} \sim 10^{-29}$ g cm^{-3}, as implied by recent cosmological data. As remarked by Linde, in grand unified theories (e.g., in the $SU(5)$ Coleman-Weinberg theory [35]) this value of the vacuum energy density is obtained as a result of a series of phase transitions. Other theories having the cosmological constant (and, therefore, vacuum energy density) decreasing with time are also discussed in the literature (e.g., a scale-covariant theory of fundamental interactions [36] in which $\Lambda \propto t^{-2}$). In ref. [7] in which $A(t)$ is related to the quantum one-loop correction terms as $\rho \sim -p_5 \sim A^{-5}$, the problem is surmounted by demanding that $A(t)$ be constant with a value cancelling out the contribution from the 5D cosmological constant, producing a zero effective 4D one.

As usually done in standard inflationary models [34], when inflation ends, the cosmological constant Λ is omitted in the field equations (3.2)-(3.4), and subsequent evolution is described by a Friedmann-type hot universe model. One may also think that the cosmological constant is contained (through vacuum energy density) as part of the off-shell matter energy density. This may have a reasonable basis, since one notes that the off-shell matter energy density with temperature dependence $\sim T^5$ [16] which at $T \sim 10^{28}$ K ($= 10^{15}$ GeV) is equal to 10^{75} GeV^4 can be consistently represented by

$$\rho^{'} = 10^{75} \left(\frac{T}{10^{28}}\right)^5 \mathrm{GeV}^4;$$

it then follows from this formula that $\rho^{'}$ takes on the value 10^{-4} GeV^4 $\sim 10^{14}$ g cm^{-3}, which is a typical energy density of radiation-like matter at $T \sim 2{\cdot}10^{12}$ K $\simeq 150$ MeV, at the same temperature, implying the possibility of a phase transition. Such a phase transition is briefly analyzed in Section 5.

4.2 Off-shell matter-dominated era

In an off-shell matter-dominated era the universe is filled with an off-shell fluid having the equation of state (3.10) with

$$\Gamma = \begin{cases} 5/4, & \sigma = -1, \\ 3/2, & \sigma = 1. \end{cases}$$

In this case, as discussed below, we omit Λ in the field equations (3.2)-(3.4). Moreover, we can also omit the spacial curvature k which is negligible at high energy densities. It then follows from Eq. (4.6) with zero r.h.s. (this also follows from (4.9) for small $\sqrt{\Lambda}t$) that

$$\tilde{R} = C_1' + C_2' t, \tag{4.18}$$

so that

for $\sigma = -1$,

$$R^{7/3} = C_1'^{-} + C_2'^{-} t, \tag{4.19}$$

for $\sigma = 1$,

$$R^{5/3} = C_1'^{+} + C_2'^{+} t. \tag{4.20}$$

4.3 On-shell matter-dominated era

In an on-shell matter-dominated era the universe is filled with a standard on-shell radiation having the equation of state $p = 1/3\rho$, and, as the universe expands and cools down, with dust matter with $p \approx 0$.

For the universe filled with radiation, we omit both Λ and k in the corresponding equation (4.5), which then has the solution

$$R^2 = C_1'^{0} + C_2'^{0} t. \tag{4.21}$$

Since Eqs. (4.19),(4.20) and (4.21) are obtained from (3.2)-(3.4) with $p_5 = \sigma p$, and $p_5 = 0$, respectively, the three Eqs. (4.19)-(4.21) can be unified in one equation, as follows:

$$R^{2-\sigma\alpha/3} = C_1'^{\sigma} + C_2'^{\sigma} t, \tag{4.22}$$

where

$$\alpha = \begin{cases} 1, & T \to \infty, \\ 0, & T \to 0, \end{cases}$$

according to (see Eqs. (A.9),(A.11) of Appendix)

$$p_5 = \begin{cases} \sigma p, & T \to \infty, \\ 0, & T \to 0. \end{cases}$$

For the universe filled with dust matter, we omit Λ alone in Eq. (4.5); this equation with zero r.h.s. has the solution

$$R^2 = C_1^{\prime\prime 0} + C_2^{\prime\prime 0} t - kt^2. \tag{4.23}$$

Two possible scenarios of evolution of the universe described by these equations exist.

First, α in Eq. (4.22) is a smooth function of T. As the universe expands and its temperature decreases, $\alpha \to 0$, so that in both cases ($\sigma = \pm 1$) Eq. (4.22) passes over smoothly to Eq. (4.21) which goes over to Eq. (4.23) at lower temperatures. That is, the universe passes smoothly from an off-shell matter-dominated era to on-shell one. In this case the rate of expansion is a smooth function of temperature (and therefore the radius of the universe) as well.

Second, α is not a smooth function of T, nor may it be a function of T at all. Since at some value of R (and therefore T) Eq. (4.19) (or (4.20)) goes over to Eq. (4.21), and the powers of R in the corresponding equations do not coincide, passage from an off-shell matter-dominated era to on-shell one occurs as a *phase transition.* In this case the rate of expansion does not change in either case of $\sigma = -1$ or $\sigma = 1$, as we shall see below).

We consider the second scenario to be more realistic one, since a passage from the off-shell sector (a sector of relativistic mass distributions [15, 16]) to on-shell one is probably a phase transition[6], as discussed in ref. [20] in the case of a relativistic Bose gas.

We now wish to discuss this phase transition in general features.

5 Phase transition from off-shell matter-dominated era to on-shell one

Using Eqs. (4.19)-(4.21), we obtain the relations representing continuity of R at t_0, where t_0 is the moment of cosmic time at which the phase transition occurs:

$(\sigma = -1) \to (\sigma = 0)$,

$$(C_1^{\prime -} + C_2^{\prime -} t_0)^{3/7} = (C_1^{\prime 0} + C_2^{\prime 0} t_0)^{1/2}, \tag{5.1}$$

$(\sigma = 1) \to (\sigma = 0)$,

$$(C_1^{\prime +} + C_2^{\prime +} t_0)^{3/5} = (C_1^{\prime 0} + C_2^{\prime 0} t_0)^{1/2}. \tag{5.2}$$

We shall study the effect of the phase transition on the expansion rate. We restrict

[6] Another possibility is a smooth Galilean limit $c \to \infty$ [37].

our consideration to the case where the phase transition occurs smoothly and adiabatically[7].

It follows from Eqs. (3.2)-(3.4),(3.6),(3.10), through the exclusion of Λ, that the following equations for R in terms of ρ hold,

$$\dot{R}^2 = \frac{1}{2}\ddot{R}R + \rho\left[\frac{\Gamma-1}{2}\left(1+\frac{\sigma}{3}\right)+\frac{1}{3}\right] - k, \tag{5.3}$$

which reduces, in the corresponding cases, to:

for $\sigma=-1,\ \Gamma=5/4$,

$$\dot{R}^2 = \frac{1}{2}\ddot{R}R + \frac{5}{12}\rho R^2 - k, \tag{5.4}$$

for $\sigma=1,\ \Gamma=3/2$,

$$\dot{R}^2 = \frac{1}{2}\ddot{R}R + \frac{2}{3}\rho R^2 - k, \tag{5.5}$$

for $\sigma=0,\ \Gamma=4/3$ (standard case),

$$\dot{R}^2 = \frac{1}{2}\ddot{R}R + \frac{1}{2}\rho R^2 - k. \tag{5.6}$$

A smooth transition occurs at the constant pressure. Using the corresponding equations of state $\rho = p/(\Gamma-1)$ and Eqs. (5.4)-(5.6) (in which we neglect k), we find the following relations which represent equality of pressure in the corresponding phases (one simply equates the quantities $(\Gamma-1)\rho R^2(t_0)$) :

$(\sigma=-1)\rightarrow(\sigma=0)$,

$$\frac{9}{49}\left(C_2'^{-}\right)^2 (C_1'^{-} + C_2'^{-}t_0)^{-8/7} = \frac{1}{4}\left(C_2'^{0}\right)^2 (C_1'^{0} + C_2'^{0}t_0)^{-1}, \tag{5.7}$$

$(\sigma=1)\rightarrow(\sigma=0)$,

$$\frac{9}{25}\left(C_2'^{+}\right)^2 (C_1'^{+} + C_2'^{+}t_0)^{-4/5} = \frac{1}{4}\left(C_2'^{0}\right)^2 (C_1'^{0} + C_2'^{0}t_0)^{-1}. \tag{5.8}$$

Calculation of $\dot{R}(t_0)$ gives, respectively,

for $\sigma=-1$,

$$\frac{3}{7}C_2'^{-}(C_1'^{-} + C_2'^{-}t_0)^{-4/7}, \tag{5.9}$$

for $\sigma=1$,

$$\frac{3}{5}C_2'^{+}(C_1'^{+} + C_2'^{+}t_0)^{-2/5}, \tag{5.10}$$

[7] More general cases of a cosmological phase transition are considered in ref. [38] on the example of a hadronic matter–the quark-gluon plasma phase transition.

for $\sigma = 0$,

$$\frac{1}{2} C_2'^0 (C_1'^0 + C_2'^0 t_0)^{-1/2}. \tag{5.11}$$

Comparison of Eqs. (5.9),(5.10) (squared) with Eq. (5.11) (squared), using Eqs. (5.7) and (5.8), shows that $\dot{R}(t_0)$ in both $\sigma = -1$ and $\sigma = 1$ phases coincide with $\dot{R}(t_0)$ in the $\sigma = 0$ phase; since $R(t_0)$ is the same for the three, we conclude that the rate of expansion,

$$H \equiv \frac{\dot{R}}{R}, \tag{5.12}$$

does not change in either case of the $(\sigma = -1) \rightarrow (\sigma = 0)$ or $(\sigma = 1) \rightarrow (\sigma = 0)$ phase transitions. This observation suggests that the phase transition should be sufficiently smooth (second order). Although a first order phase transition might be preferable for cosmological implications, due to the fluctuations which are generated at the transition[8], experimental indications on the order of this phase transition are still absent. Indeed, cosmological phase transition at $T_c \sim 150$ MeV is normally associated with the transition from a strongly interacting hadronic phase to a weakly interacting quark-gluon plasma phase [38, 44]. Presently available lattice data on $SU(N)$ pure gauge theory lattice simulations indicate that a phase transition to a weakly interacting phase is of apparently first order for $SU(3)$ and second order for $SU(2)$ theory [45]. In ref. [46], however, it is argued that the apparent first order nature of the transition in the case of $SU(3)$ pure gauge theory may well be a lattice artefact. Moreover, there are indications from lattice QCD calculations that when fermions are included, the phase transition may be of second or higher order [47]. In this case, as remarked by Ornik and Weiner [44], the phase transition would be hardly distinguishable from a situation in which no phase transition would have taken place (radiation-dominated universe alone).

6 "Generalized" entropy and the behavior of A in an on-shell matter-dominated era

As we have seen in Section 3, the expansion of the universe with the line element (2.17) is not adiabatic, due to the presence of time-dependent A in the equation (3.5) for energy-momentum conservation. Rewriting this equation in the form

$$\left[R^3 A \rho\right]^{\bullet} + Ap\dot{R^3} - p_5 R^3 \dot{A} = 0, \tag{6.1}$$

[8] The fluctuations could not directly affect galaxy formation, since the horizon size at the time of the transition is on a planetary scale [40]. It has been demonstrated [41] that they could produce planetary mass black holes; these black holes could provide a possible explanation for the dark matter of the universe and even be seeds in galaxy formation [42, 43].

we see that the universe can be characterized by the "first law"

$$d(AE) = ATd\tilde{S} - ApdV + p_5 VdA, \tag{6.2}$$

so that, in view of (6.1),(6.2) and $V \sim R^3$, the "generalized" entropy $\tilde{S}$ is conserved:

$$\frac{d\tilde{S}}{dt} = 0. \tag{6.3}$$

It follows from (6.2) and genuine first law

$$dE = TdS - pdV \tag{6.4}$$

that the thermodynamic entropy are related to the "generalized" one as follows:

$$dS = d\tilde{S} - \frac{E'}{T}\frac{dA}{A}, \tag{6.5}$$

where $E' \equiv E - p_5 V = \rho' V$, and

$$\rho' \equiv \rho - p_5$$

is the "reduced" energy density [14]. Hence, in view of (6.3),

$$\frac{dS}{dt} = -\frac{E'}{T}\frac{1}{A}\frac{dA}{dt}. \tag{6.6}$$

One sees that the sign of $\frac{dS}{dt}$ is determined by the sign of $-\frac{1}{A}\frac{dA}{dt}$. Note that, as $A \to$ const, it follows from (6.5) that $S = \tilde{S} + \text{const}$.

In general, the time dependence of A can be derived from Eqs. (3.2)-(3.4), (3.6), (3.10), provided that the corresponding time dependence of R is known. We shall restrict ourselves to the on-shell matter-dominated universe (similar consideration for the vacuum- and the off-shell matter-dominated eras does not seem to present a difficulty). For the universe filled with radiation-like matter, Eqs. (3.2)-(3.4) take on the form

$$\frac{\dot{R}^2}{R^2} + \frac{\dot{R}\dot{A}}{RA} = \frac{1}{3}\rho = p, \tag{6.7}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{2\dot{R}\dot{A}}{RA} + \frac{\ddot{A}}{A} = -p, \tag{6.8}$$

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = 0. \tag{6.9}$$

It follows from Eq. (6.9) that

$$R^2 = C_1'^0 + C_2'^0 t,$$

in agreement with (4.21). Summing up Eqs. (6.7) and (6.8), taking into account (6.9), gives

$$\frac{3\dot{R}\dot{A}}{RA}+\frac{\ddot{A}}{A}=0. \tag{6.10}$$

This equation has the solution

$$\dot{A}=\frac{C}{R^3}, \quad C=\mathrm{const}; \tag{6.11}$$

hence

$$\begin{aligned} A &= C\int\frac{dt}{R^3} = C\int\frac{dt}{(C_1'^0+C_2'^0t)^{3/2}} \\ &= \frac{-2C/C_2'^0}{(C_1'^0+C_2'^0t)^{1/2}}+\beta = \frac{\alpha}{R}+\beta, \quad \alpha,\beta=\mathrm{const}. \end{aligned} \tag{6.12}$$

Therefore

$$-\frac{1}{A}\frac{dA}{dt}=\frac{\dot{R}}{R(1+\frac{\beta}{\alpha}R)}. \tag{6.13}$$

One sees that, since $\dot{R}>0$, if α and β are of the same sign, or if $\beta=0$, then $-\frac{1}{A}\frac{dA}{dt}>0$, and therefore $\frac{dS}{dt}>0$, in view of (6.6).

For the dust universe, it follows from Eqs. (3.3),(3.4) with zero r.h.s. that

$$\frac{\ddot{R}}{R}+\frac{2\dot{R}\dot{A}}{RA}+\frac{\ddot{A}}{A}=0, \tag{6.14}$$

which reduces to the equation

$$\ddot{(AR)}=0, \tag{6.15}$$

which has the solution (taking into account (4.23))

$$A=\frac{a+bt}{R}=\frac{a+bt}{\sqrt{C_1''^0+C_2''^0t-kt^2}}, \quad a,b=\mathrm{const}. \tag{6.16}$$

The substitution

$$\rho=\frac{3\gamma}{AR^3}, \quad \gamma=\mathrm{const} \tag{6.17}$$

(which follows from (3.14)) in the r.h.s. of Eq. (3.2) (in which we omit Λ, as usual) yields, with the help of Eq. (3.4) with zero r.h.s.,

$$\dot{R}\dot{A}-A\ddot{R}=\frac{\gamma}{R^2}. \tag{6.18}$$

One can then find that Eqs. (6.16) and (6.18) are compatible if

$$\left\{\begin{array}{rcl} k &=& 0, \\ bC_2''^0 &=& 2\gamma, \end{array}\right. \tag{6.19}$$

$$\begin{cases} k & \neq & 0, \\ a & = & \gamma k, \\ bC_2''^0 & = & 0, \end{cases} \tag{6.20}$$

and the same relations (6.19),(6.20) with $C_1''^0 = 0$. Moreover, the solutions (4.23) for R and (6.16) for A should be matched[9] with the corresponding solutions (4.21),(6.12) for the radiation-like universe, which we rewrite here:

$$\begin{cases} R & = & \sqrt{C_1'^0 + C_2'^0 t}, \\ A & = & \frac{\alpha}{\sqrt{C_1'^0 + C_2'^0 t}} + \beta. \end{cases} \tag{6.21}$$

Thus, one is left with the following solutions for the dust universe:

for $k = 0$,

$$\begin{cases} R & = & \sqrt{C_1''^0 + C_2''^0 t}, \\ A & = & \frac{a + 2\gamma t / C_2''^0}{\sqrt{C_1''^0 + C_2''^0 t}}, \end{cases} \tag{6.22}$$

for $k = 1$,

$$\begin{cases} R & = & \sqrt{C_1''^0 + C_2''^0 t - t^2}, \\ A & = & \frac{\gamma}{\sqrt{C_1''^0 + C_2''^0 t - t^2}}, \end{cases} \tag{6.23}$$

for $k = -1$,

$$\begin{cases} R & = & \sqrt{C_1''^0 + t^2}, \\ A & = & \frac{bt - \gamma}{\sqrt{C_1''^0 + t^2}}. \end{cases} \tag{6.24}$$

6.1 Flat dust universe

Consider first the case of the flat dust universe. It follows from (6.22) that, as $t \to \infty$,

$$A \sim R \sim t^{1/2}. \tag{6.25}$$

Therefore, $-\frac{1}{A}\frac{dA}{dt} = -\frac{1}{R}\frac{dR}{dt} = -H(t) < 0$, so that the entropy is a decreasing function of cosmic time. In this case, as seen in Eq. (2.21), if $p_\tau < 0$,

$$\frac{dS}{d\tau} = \frac{dt}{d\tau}\frac{dS}{dt} > 0, \tag{6.26}$$

i.e., the entropy is a growing function of *historical time*; if we moreover take α and β in (6.12) of the opposite sign, we will have, in view of (6.13), $\frac{dS}{d\tau} > 0$ during all the on-shell matter-dominated era.

It is seen in Eq. (6.25) that for the model (6.22), the limit $A \to$ const is absent, i.e., it does not go over to the standard 4D cosmological flat model, but rather represents the model (2.5) with $e^{\bar{\omega}} = e^{\bar{\mu}} = t$.

[9] By matching we mean continuity of a function and its first derivative.

6.2 Closed dust universe

Now we turn to the case of the closed dust universe. As seen in Eqs. (6.23), the universe expands until the moment

$$t_0 = \frac{C_2''^0}{2}, \tag{6.27}$$

reaching the maximal radius, and then begins to contract. For this model,

$$\frac{dA}{dt} = \frac{\gamma(t - \frac{C_2''^0}{2})}{(C_1''^0 + C_2''^0 t - t^2)^{3/2}} = \frac{\gamma(t - \frac{C_2''^0}{2})}{R^3}, \tag{6.28}$$

so that, as $t \to t_0$, $\frac{dA}{dt} \simeq 0$, i.e., $A \simeq$ const. Therefore, in this limit the line element (2.17) goes over to the standard 4D one, (2.23), and the 5D gravitational field equations become indistinguishable from the standard 4D Einstein equations.

Since $A = \gamma/R$,

$$-\frac{1}{A}\frac{dA}{dt} = \frac{1}{R}\frac{dR}{dt} = H(t) = \frac{\frac{C_2''^0}{2} - t}{C_1''^0 + C_2''^0 t - t^2} = \frac{\frac{C_2''^0}{2} - t}{R^2}. \tag{6.29}$$

We see that, as $t \le C_2''^0/2$, $-\frac{1}{A}\frac{dA}{dt} \ge 0$, and therefore $\frac{dS}{dt} \ge 0$, via (6.6); hence, by virtue of (6.14),(6.28), for the closed universe the entropy increases during the expansion in the whole on-shell matter-dominated era, reaching its maximum at $t = t_0$, where $\frac{dS}{dt} = 0$.

Rewriting (6.23) in the form

$$\left\{ \begin{array}{lcl} R & = & \sqrt{C_1''^0 + (C_2''^0)^2/4 - \tilde{t}^2}, \\ A & = & \gamma/R, \end{array} \right. \tag{6.30}$$

where

$$\tilde{t} \equiv C_2''^0/2 - t, \tag{6.31}$$

we see that the model possesses explicit $\tilde{t}$-reversal. Since

$$\frac{dS}{d\tilde{t}} = -\frac{dS}{dt}, \tag{6.32}$$

one sees that, for $t > t_0$, when the universe contracts, the entropy is a growing function of $\tilde{t}$, as seen in Eqs. (6.29),(6.32).

Let us write down the formula which is valid for the closed universe and follows from Eq. (6.6) (with $E' = E$ for the on-shell matter-dominated universe) and $A \sim 1/R$:

$$\frac{dS}{dt} = \frac{E}{T}H(t). \tag{6.33}$$

Note also that for $A \sim 1/R$, the energy-momentum conservation (3.5) yields the relation (3.15) for the dust universe. Indeed, in the case of local $O(4,1)$ invariance of the (x^μ, τ) manifold, it follows from (3.11) with $A \sim 1/R$ that $R^{4\Gamma-2}\rho = \text{const}$; for $\Gamma = 5/4$ the latter reduces to (3.15). Similarly, in the $O(3,2)$ case, one obtains from (3.13) $R^{2\Gamma}\rho = \text{const}$, which again reduces to (3.15) for $\Gamma = 3/2$.

6.3 Open dust universe

For the open dust universe, as seen in Eqs. (6.24), as $t \to \infty$,

$$R \simeq t, \quad A \simeq b - \gamma/t \to b = \text{const}, \tag{6.34}$$

so that this model tends asymptotically to the standard 4D cosmological open model, for which $R \simeq t$ at large t [39]. It follows from (6.24) that, as $t \to \infty$,

$$-\frac{1}{A}\frac{dA}{dt} = -\frac{bC_1^{\prime\prime 0} + \gamma t}{(C_1^{\prime\prime 0} + t^2)^{3/2}} \to -\frac{\gamma}{t^2} < 0, \tag{6.35}$$

since A and γ are of the same sign, in view of (6.17). Thus, for the open dust universe, the entropy is a decreasing function of cosmic time, but it can be a growing function of historical time, if $p_\tau < 0$, similarly to the case of the flat dust universe.

7 Concluding remarks

We have considered 5D Kaluza-Klein type cosmological model with the fifth coordinate being an invariant historical time τ. We have derived a complete cosmological inflationary scenario for such a model which distinguishes between vacuum-, off-shell matter-, and on-shell matter-dominated eras as the solutions of the corresponding 5D gravitational field equations. According to this scenario, the passage from the off-shell matter-dominated era to the on-shell one occurs, probably as a phase transition. We have studied the effect of this phase transition on the expansion rate and found that it does not change in either case of local $O(4,1)$- or $O(3,2)$-invariance of the extended (x^μ, τ)-manifold.

In contrast to the standard cosmological model in which the expansion of the universe is adiabatic, $dS/dt = 0$ [18, 19], the model considered here does not expand adiabatically; the thermodynamic entropy is a growing function of cosmic time for the closed universe, and can be a growing function of historical time for the open and the flat universes.

We have obtained a complete solution of the 5D gravitational field equations for the on-shell matter-dominated universe. We have shown that the 5D open and closed universes tend asymptotically to the corresponding standard 4D cosmological models, in contrast to the 5D flat universe which does not have such a corresponding limit.

The question of the choice of the source term in the form containing p_5 (like (2.3)) has received attention in the recent literature. Mann and Vincent [21] have considered the case of local $O(4,1)$ invariance of 5D manifold and used the source term with p_5 involved. The effective 4D equation of state obtained by them from the vacuum solution of the 5D equations, $\rho = 3p,\ p_5 = 0$, is essentially the one used in ref. [7] with the quantum one-loop correction terms ($\rho \sim -p_5 \sim A^{-5}$) neglected in comparison with the classical 5D radiation term. Wesson [29] has considered the 5D version of 4D field equations, $R_{\alpha\beta} = T_{\alpha\beta}$, with the source term $T_{\alpha\beta}$ in the form (3.9) but only solved $R_{\alpha\beta} = 0$. Later on, Wesson and independently Ponce de Leon [48] suggested that the 5D field equations may be just $R_{\alpha\beta} = 0$, and the extra terms which appears on the left-hand sides of the 5D equations $R_{\alpha\beta} = 0$ may correspond to the terms involving matter parameters (like the density and pressure) which appear on the right-hand sides of the 4D equations $R_{\mu\nu} = T_{\mu\nu}$. More recently, Ponce de Leon and Wesson [26] have shown that a 5D theory with no source can be cast into the form of a 4D theory with a source, in the three-dimensionally symmetric case. In this case (when the metric is independent of the extra coordinate, which is the case we consider in our paper), as they have shown, the equation of state has the form of that of radiation-like matter, $p = \rho/3$. Hence, it is not possible, in the case when the source term is obtained from the geometry of the higher dimensional theory alone, to achieve the equation of state of a strongly interacting phase (e.g., $p = \rho/4$, which has certain experimental evidence [49]), as we have obtained with an explicit source term, nor to describe the inflationary epoch.

By rewriting the equation of the energy-momentum conservation, (3.5), in the form

$$\dot{\rho} + \left(3\frac{\dot{R}}{R} + \frac{\dot{A}}{A}\right)(\rho + p) - \frac{\dot{A}}{A}(p + p_5) = 0,$$

one sees that the question of the presence of p_5 in the source term for the field equations (3.2)-(3.4) is associated with the question of whether or not the cosmological fluid is ideal. In the case of local $O(4,1)$ invariance of (x^μ, τ) manifold, when $p_5 = -p$, the latter equation reduces to

$$\dot{\rho} + 3\frac{\dot{R}'}{R'}(\rho + p) = 0,$$

which is the standard equation for ideal cosmological fluid expanding non-adiabatically due to the scale factor $R' = RA^{1/3}$. In the other cases, $p_5 = p$ ($O(3,2)$) or $p_5 = 0$ [21], the cosmological fluid is not ideal since the energy-momentum tensor has an anisotropic pressure. Note that, as follows from (3.5), in the case of local $O(3,2)$ invariance of (x^μ, τ) manifold, the cosmological fluid is ideal and expands *adiabatically* if $p_5 = \rho$, implying the equality of the energy and mass densities, as for standard radiation-like matter. Evolution of the universe filled with such a fluid will be the subject of subsequent study.

Appendix

In this Appendix we review briefly a five-dimensional theory of gravitation with τ as a fifth coordinate. Full consideration will be given elsewhere [50]. We shall proceed in a way similar to the one chosen by Ma [51] within a different framework. Suppose that the fifth-dimension subspace is homogeneous and without coupling to the other coordinate, i.e., it is a maximally symmetric subspace of the 5D space [30]. With this assumption, the 5D metric becomes [30]

$$^{(5)}ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = g_{\mu\nu}(x^\rho)dx^\mu dx^\nu + g_{55}(x^\rho)d\tau^2, \tag{A.1}$$

$$\alpha, \beta = 0, 1, 2, 3, 5; \quad \mu, \nu, \rho = 0, 1, 2, 3.$$

The line element in the hypersurface $\tau = \text{const}$ is that of the 4D space-time:

$$^{(4)}ds^2 = g_{\mu\nu}(x^\rho)dx^\mu dx^\nu.$$

Since $g_{\alpha\beta}g^{\alpha\gamma} = \delta^\gamma_\beta$ implies $g_{\mu\nu}g^{\mu\rho} = \delta^\rho_\nu$ and $g_{55}g^{55} = 1$, the only non-zero components of the 5D Christoffel symbol $^{(5)}\Gamma^\alpha_{\beta\gamma}$ are

$$^{(5)}\Gamma^\mu_{\nu\rho} = {}^{(4)}\Gamma^\mu_{\nu\rho}, \quad {}^{(5)}\Gamma^5_{5\nu} = \frac{1}{2}g_{55}\frac{\partial g_{55}}{\partial x^\nu}, \quad {}^{(5)}\Gamma^\mu_{55} = -\frac{1}{2}g^{\mu\nu}\frac{\partial g_{55}}{\partial x^\nu}. \tag{A.2}$$

The non-vanishing components of the 5D Ricci tensor and the 5D scalar curvature are, respectively,

$$\begin{aligned} ^{(5)}R_{\mu\nu} &= {}^{(4)}R_{\mu\nu} + \frac{\partial \Gamma^5_{\mu 5}}{\partial x^\nu} + \Gamma^5_{\mu 5}\Gamma^5_{\nu 5} - \Gamma^\rho_{\mu\nu}\Gamma^5_{\rho 5}, \\ ^{(5)}R_{55} &= -\frac{\partial \Gamma^\mu_{55}}{\partial x^\mu} + \Gamma^\mu_{55}\Gamma^5_{\mu 5} - \Gamma^\mu_{55}\Gamma^\nu_{\nu\mu}; \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} ^{(5)}R &= {}^{(4)}R + g^{\mu\nu}\left(\frac{\partial \Gamma^5_{\mu 5}}{\partial x^\nu} + \Gamma^5_{\mu 5}\Gamma^5_{\nu 5} - \Gamma^\rho_{\mu\nu}\Gamma^5_{\rho 5}\right) \\ &\quad + g^{55}\left(-\frac{\partial \Gamma^\mu_{55}}{\partial x^\mu} + \Gamma^\mu_{55}\Gamma^5_{\mu 5} - \Gamma^\mu_{55}\Gamma^\nu_{\nu\mu}\right), \end{aligned} \tag{A.4}$$

where we have dropped the indices on the left of the Christoffel symbol.

The D-dimensional gravitational field equations read

$$^{(D)}R_{mn} - \frac{1}{2}g_{mn}\,{}^{(D)}R = 8\pi G\,{}^{(D)}T_{mn}, \quad m, n = 0, 1, \ldots, D-1.$$

Contracting with g^{mn} gives

$$^{(D)}R = -\frac{16\pi G}{D-2}\,{}^{(D)}T^k_k,$$

and therefore, the field equations can be rewritten as

$$ {}^{(D)}R_{mn} = 8\pi G \left({}^{(D)}T_{mn} - \frac{1}{D-2} g_{mn} \, {}^{(D)}T^{k}_{k} \right). $$

Thus, the 5D gravitational field equations read

$$ {}^{(5)}R_{\alpha\beta} = 8\pi G \left({}^{(5)}T_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} \, {}^{(5)}T^{\lambda}_{\lambda} \right), \tag{A.5} $$

and can be written in the following component form:

$$ \begin{aligned} {}^{(5)}R_{\mu\nu} &= 8\pi G \left({}^{(5)}T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, {}^{(5)}T^{\rho}_{\rho} \right) + \frac{4\pi G}{3} g_{\mu\nu} \left({}^{(5)}T^{\rho}_{\rho} - 2 \, {}^{(5)}T^{5}_{5} \right), \\ {}^{(5)}R_{55} &= \frac{8\pi G}{3} g_{55} \left({}^{(5)}T^{\rho}_{\rho} - 2 \, {}^{(5)}T^{5}_{5} \right), \quad {}^{(5)}R_{\mu 5} = 0, \quad \rho = 0,1,2,3. \end{aligned} \tag{A.6} $$

The natural requirement on the 5D theory we are dealing with is to contain the Einstein 4D theory of gravitation. Therefore, the 5D field equations (A.6) should reduce to the Einstein 4D field equations in the standard case. Such a case is $g_{55} =$ const, i.e., no curvature associated with τ direction. In this case we have

$$ \Gamma^{5}_{\mu 5} = \Gamma^{\mu}_{55} = 0, \quad {}^{(5)}R_{\mu\nu} = {}^{(4)}R_{\mu\nu}, \quad {}^{(5)}R_{55} = 0, \quad {}^{(5)}R = {}^{(4)}R $$

and

$$ \begin{aligned} {}^{(4)}R_{\mu\nu} &= 8\pi G \left({}^{(5)}T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, {}^{(5)}T^{\rho}_{\rho} \right), \\ {}^{(5)}T_{\mu 5} &= 0, \quad 2 \, {}^{(5)}T^{5}_{5} - {}^{(5)}T^{\rho}_{\rho} = 0. \end{aligned} \tag{A.7} $$

We require that (A.7) are exactly the Einstein equations; then [51]

$$ {}^{(5)}T_{\mu\nu} = {}^{(4)}T_{\mu\nu}, \quad {}^{(5)}T_{\mu 5} = 0, \quad {}^{(5)}T^{5}_{5} = \frac{1}{2} \, {}^{(4)}T^{\rho}_{\rho}. \tag{A.8} $$

The quantity ${}^{(5)}T^{5}_{5} = \frac{1}{2} \, {}^{(4)}T^{\rho}_{\rho}$ was calculated in ref. [14]:

$$ {}^{(5)}T^{5}_{5} \equiv p_5 = \sigma \mu_K \kappa, \tag{A.9} $$

where κ is the density of the generalized Hamiltonian per unit comoving three-volume (actually associated with the density of the variable mass squared), μ_K is the mass potential in relativistic ensemble [17], and $\sigma = \pm 1$, according to invariance group of an extended (x^{μ}, τ) manifold which could be $O(3,2)$ or $O(4,1)$.

Thus, the 5D gravitational field equations (A.5) with the source term

$$ {}^{(5)}T_{\alpha\beta} = \left({}^{(4)}T_{\mu\nu}, \; p_5 \right) \tag{A.10} $$

reduce to the 4D Einstein equations in the assumption of no curvature in τ direction.

As found in [14],

$$\mu_K \kappa \simeq \begin{cases} p, & T \to \infty, \\ 0, & T \to \ 0. \end{cases} \tag{A.11}$$

Therefore, as $T \to 0$,

$$^{(5)}T_{\alpha\beta} = \left(\ ^{(4)}T_{\mu\nu}, 0 \right), \tag{A.12}$$

where

$$^{(4)}T_{\mu\nu} = (p+\rho) u_\mu u_\nu - p g_{\mu\nu}, \quad u^\rho u_\rho = 1 \tag{A.13}$$

is the 4D energy-momentum tensor;
as $T \to \infty$,

$$^{(5)}T_{\alpha\beta} = \left(\ ^{(4)}T_{\mu\nu}, \sigma p \right),$$

i.e.,

$$^{(5)}T_{\alpha\beta} = (p+\rho) u_\alpha u_\beta - p g_{\alpha\beta}, \quad u^\lambda u_\lambda = 1, \tag{A.14}$$

$$g_{\alpha\beta} = (g_{\mu\nu}, \ \sigma), \quad \sigma = \pm 1,$$

represents the generalized 5D energy-momentum tensor.

The 5D geodesic equations

$$\frac{d^2 x^\alpha}{ds^2} + \ ^{(5)}\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0 \tag{A.15}$$

reduce, via (A.2), to

$$\frac{d^2 x^\mu}{ds^2} + \ ^{(5)}\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} + \ ^{(5)}\Gamma^\mu_{55} \left(\frac{d\tau}{ds} \right)^2 = 0, \tag{A.16}$$

$$\frac{d^2 \tau}{ds^2} + 2 \ ^{(5)}\Gamma^5_{\mu 5} \frac{d\tau}{ds} \frac{dx^\mu}{ds} = 0. \tag{A.17}$$

In the standard case g_{55} =const, $^{(5)}\Gamma^5_{\mu 5} = \ ^{(5)}\Gamma^\mu_{55} = 0$, and Eqs. (A.16),(A.17) take on the form

$$\frac{d^2 x^\mu}{ds^2} + \ ^{(5)}\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0, \tag{A.18}$$

$$\frac{d^2 \tau}{ds^2} = 0. \tag{A.19}$$

It then follows that $\tau = as + b, \ \ a, b = \text{const}, \ \ a \neq 0$, and therefore, (A.18) reduces (since $^{(5)}\Gamma^\mu_{\nu\rho} = \ ^{(4)}\Gamma^\mu_{\nu\rho}$) to

$$\frac{d^2 x^\mu}{d\tau^2} + \ ^{(4)}\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \tag{A.20}$$

the standard 4D geodesic equations for the metric

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{A.21}$$

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